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Divergence and flutter instabilities of a cantilever beam subjected to a terminal dynamic moment

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Interest of the work

Since the 1950's, scientists have studied the stability of elastic systems. Depending on the nature of the loading, elastic systems exhibit divergence and flutter modes of instability. Divergence is a static instability which occurs when the static aerodynamic effects counteract the torsional stiffness of the structure. In this case, the complex conjugate roots associated with the fundamental frequency becomes equal to zero. Flutter is a dynamic aeroelastic instability characterized by sustained oscillation of structure arising from interaction between the elastic, inertial and aerodynamic forces acting on the body. In this case, the roots associated with two consecutive natural frequencies become equal.

Many studies of cantilevered structures with a static moment applied to its free end have been conducted, but the stability characteristics of a cantilever beam with a terminal dynamic moment did not have been considered prior to the article. That's why the authors study the stability characteristics of a cantilever beam subjected to a dynamic moment at its free end using both theory and experiments. To do this, they assumed the terminal moment as being proportional to the slope or curvature of the beam at some points along its length.

Resolution process

The structure considered is the cantilevered Euler-Bernoulli beam of length L and uniform cross-sectional area A which its free end is subjected to a dynamic bending moment M. This bending moment is produced by an actuator mounted at the free end of mass m and mass moment of inertia J. They made the hypothesis of small deformations so the equation of motion of the beam and its boundary conditions are:

$$EIy'''' + \rho A\ddot{y} = 0$$

$$y(0,t) = 0, \qquad y'(0,t) = 0, \qquad EIy''(L,t) + J\ddot{y}' = M, \qquad EIy'''(L,t) = m\ddot{y}(L,t)$$

They chose to do a change of variables to have a non-dimensional equation

$$v = \frac{y}{L}, \qquad u = \frac{x}{L}, \qquad \tau = t\sqrt{\frac{EI}{\rho A L^4}}$$

$$v''''(u,\tau) + \ddot{v}(u,\tau) = 0$$

$$v(0,\tau) = 0, \qquad v'(0,\tau) = 0, \qquad v''(1,\tau) + \eta\ddot{v}'(1,\tau) = \bar{M}, \qquad v'''(1,\tau) = \mu\ddot{v}(1,\tau)$$

Where

$$\bar{M} \triangleq \frac{ML}{EI}, \qquad \mu \triangleq \frac{m}{\rho A L^3}, \qquad \eta \triangleq \frac{J}{\rho A L^3}$$

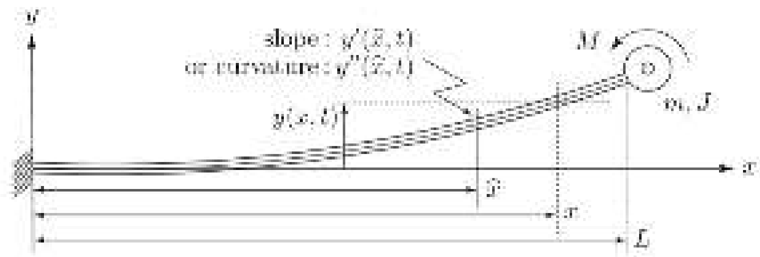

Figure 1 Cantilever beam with a tip mass and a terminal dynamic moment

Two cases were considered: The terminal moment M is proportional to the slope of the beam and the terminal moment M is proportional to the curvature of the beam.

$$M = M_s \triangleq C_s y'(\hat{x}, t) : slope - dependent$$

$$M = M_c \triangleq C_c y''(\hat{x}, t) : curvature - dependent$$

Where $\hat{x}$ is the point where the slope or the curvature is measured, and where C_s and C_c are proportional constants.

Using non-dimensional variables, we have

$$\bar{M} = \overline{M_s} \triangleq \overline{C_s} v'(\alpha, \tau), \qquad \overline{C_s} \triangleq \frac{C_s L}{EI} : slope - dependent$$

$$\bar{M} = \overline{M_c} \triangleq \overline{C_c} v''(\alpha, \tau), \qquad \overline{C_c} \triangleq \frac{C_c}{EI} : curvature - dependent$$

Where $\alpha \triangleq \frac{\hat{x}}{L} \in (0,1]$ denotates the point where the slope or the curvature is measured.

In order to solve the problem, the authors used analytical and numerical elements.

Analytical solution

The authors used a variable separation, $v(u, \tau) = U(u)T(\tau)$, to get solutions of the form

$$T(\tau) = Acos(\bar{\omega}\tau) + Bsin(\bar{\omega}\tau)$$

$$U(u) = P_1 e^{\beta u} + P_2 e^{-\beta u} + P_3 e^{i\beta u} + P_4 e^{-i\beta u}$$

Where $\bar{\omega} \triangleq \beta^2$. A and B are constants that can be obtained from initial conditions and the P's are constants that can be obtained from the boundary conditions.

The non-dimensional frequencies $\bar{\omega}$ can be obtained by solving the transcendental characteristic equations which results from the boundary conditions. Each solution of β is a unique mode shape. By solving the problem, they determined the critical stability points because the terminal moment is a boundary condition.

Numerical solution

The authors used the Galerkin method to solve the problem. They supposed the solution of the equation of motion of the beam to be of the form

$$v(u,\tau) = \sum_{i=1}^{N} a_i(\tau)\varphi_i(u)$$

Where N is the number of terms of approximation (N=10), $\varphi_i(u)$ the modes that satisfy the geometric boundary conditions and $a_i(\tau)$ the modal amplitudes. Using that, they got the equation of the problem

$$[M]\ddot{a} + [K]a = 0$$

Where $[M]$ is the mass matrix and $[K]$ is the geometric stiffness matrix.

Principle results

In order to study the instabilities of the cantilever beam due to the terminal dynamic moment, they used both methods. They plotted $\overline{\omega}$ as a function of $\overline{C_s}$ or $\overline{C_c}$ for discrete values of α, with μ=7.03 and η=0.038[1], and they considered the proportionality constants to be negative or positive.

They found that when the dynamic moment is proportional to positive slope, the first natural frequency reduces to zero and the system loses stability through divergence[2]. Also, when $\overline{C_s}$ is increased beyond the first critical value, some consecutive frequencies become complex and the system loses stability through flutter[3].

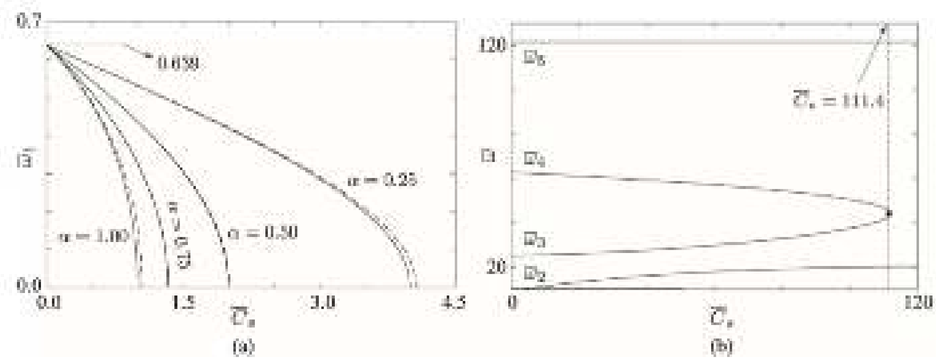

Figure 2 (a) Variation in the first frequency. (b) Variation of the first five frequencies for a=0.25

Next, when the dynamic moment is proportional to negative slope, the system loses stability through flutter as two consecutive frequencies become complex for a value of $\overline{C_s}$. Also, by plotting different values of α, they found that the critical value of $\overline{C_s}$ decreases as α increases from 0.1 to 0.3 but increases when α increases from 0.3 to 0.63. Equally, the critical value of $\overline{C_s}$ decreases as α increases from 0.64 to 0.8 but increases when α increases further. This effect is called a "destabilizing-stabilizing" effect of the location of the point from where the slope of the beam is measured. To illustrate it, they plotted a critical stability curve. We can see that for some values of $\overline{C_s}$ there is a "destabilizing-stabilizing- destabilizing-stabilizing" effect as the value of $\overline{C_s}$ intersects the curve several times.

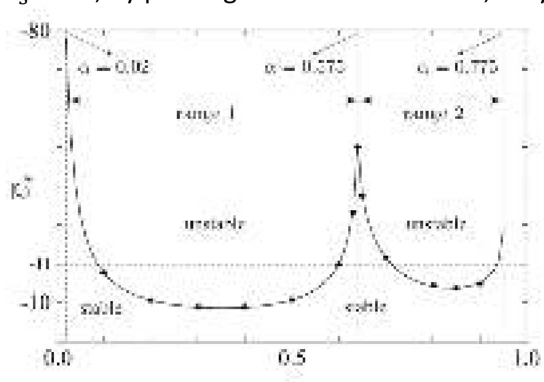

Figure 3 Critical stability curve

[1] Same values as for the experimental setup

[2] Each value of α has a critical value of $\overline{C_s}$ but the trend is the same.

[3] Couple-mode flutter via a Hamilton-Hopf bifurcation

Similar results where obtained with a dynamic moment proportional to positive or negative curvature. For positive values of $\overline{C_c}$ the system loses stability through divergence but this time the critical value of $\overline{C_c}$ is almost the same for different values of α. For negative values of $\overline{C_c}$ the system loses stability through flutter and higher values of α are associated with higher modes of flutter instability.

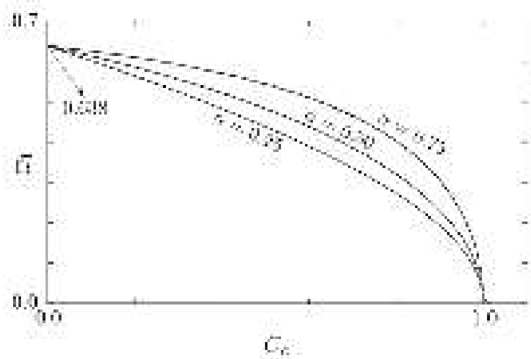

Figure 4 Variation in the first natural frequency

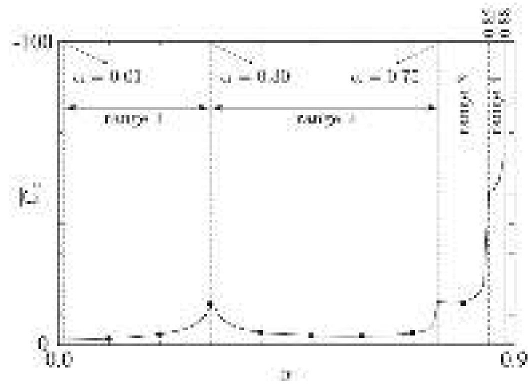

Figure 5 Critical stability curve

To verify the theoretical results, they did an experimental investigation with a steel cantilever beam which had a dynamic moment at the free end of the beam applied by a motor. Firstly, they verified that the natural frequencies matched well with those found theoretically. After that, they applied the vibrations to the system but as the system nonlinearities lead to limit cycle oscillations at the point of instability, or slightly beyond it, the results could be slightly different to the linear model. Nevertheless, they found that the results were similar to those found theoretically even though the theoretical results were for a linear model.

Table 1 Results comparison

	f^* (Hz)	C_c^* (Nm^2)
Analytical	1.168	−0.262
Numerical	1.167	−0.263
Experimental	1.33	−0.21

Conclusion

For both cases, terminal moment proportional to the slope and terminal moment proportional to the curvature, the beam loses stability through divergence when the constant of proportionality is positive and through flutter when it is negative. For the case where the terminal moment is proportional to the negative slope or negative curvature, multiple stability transitions can occur and higher modes of flutter instability are induced as the point of measurement shifts from the fixed end to the free end of the beam. The experimental work confirmed the theoretically obtained results.

Today it is important to study this kind of phenomena because they can be a cause of disaster. It appears in many engineering areas like structural engineering when studying the wind effect on the bridges and skyscrapers, in marine applications and power-plant engineering or in aviation and rocket science. A great example of catastrophe due to this kind of phenomena is the destruction of the original Tacoma Narrows Bridge as a result of aeroelastic fluttering. Therefore, further studies are needed to prevent the problems by adjusting the mass, stiffness or even the placement of mass balances.